Decomposability of representations

Prop: Every (non-0) representation of G is either irreducible or decomposable. $(|G| < \infty)$

Proof: $\varphi : G \to GL(V)$, $V \neq \{0\}$

Suppose φ not irreducible.

$\Longrightarrow \exists$ $W \leq V$ invariant subspace

$W \neq \{0\}$, $W \neq V$

Then: $\exists$ linear $T : V \to V$ such that

$T(V) \subseteq W$, $T|_W = \mathrm{Id}_W$ $\Rightarrow$ $TT = T$

Average T over φ:

$$T' := \frac{1}{|G|} \sum_{g \in G}{}' \varphi_g T \varphi_{g^{-1}} : V \to V$$

in fact, $T' \in \mathrm{Hom}_G(\varphi, \varphi)$

Claim: (1) $T'(V) \subseteq W$
(2) $T'|_W = \mathrm{Id}_W$ ✓ $\Rightarrow T'T' = T'$

$$T'(v) = \frac{1}{|G|} \sum_{g \in G} \varphi_g \overbrace{T \underbrace{\varphi_{g^{-1}}(v)}_{\in W \text{ if } v \in W}}^{\in W} \in W$$

If $v \in W$

$$= \frac{1}{|G|} \sum_{g \in G} \cancel{\varphi_g \varphi_{g^{-1}}}(v) = \frac{1}{|G|} \sum_{g \in G} v = v$$

Define: $W' := \ker T' = \{x \in V \mid T'x = 0\}$

T' morphism, $\Rightarrow W'$ invariant subspace

Claim: $\underline{V = W \oplus W'} \Rightarrow \underline{\underline{\varphi = (\varphi|_W) \oplus (\varphi|_{W'})}}$

$x \in V$

$$\Rightarrow x = \underbrace{T'(x)}_{\text{in } W} + \underbrace{(x - T'(x))}_{\text{in } W'}$$

$\Rightarrow V = W + W'$

If $x \in \underline{\underline{W}} \cap \underline{\underline{W'}} = \{0\}$ $\quad x = T'(x) = 0$

✓